

VARIATIONS OF EXTREMAL LENGTH FUNCTION ON TEICHMÜLLER SPACE

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ABSTRACT. By identifying extremal length function with energy of harmonic map from Riemann surface to \mathbb{R} -tree, we compute the second variation of extremal length function along Weil-Petersson geodesic. We show that the extremal length of any simple closed curve is a pluri-subharmonic function on Teichmüller space.

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1. INTRODUCTION

Extremal length is an important tool in the study of quasiconformal map and Teichmüller theory. The notion is due to Ahlfors and Beurling [2].

Let X be a Riemann surface. A *conformal metric* ρ on X is locally of the form $\rho(z)|dz|$ where $\rho(z) \geq 0$ is a Borel measurable function. We define the ρ -area of X by

$$\text{Area}_\rho(X) = \int_X \rho^2(z) |dz|^2.$$

Let \mathcal{S} be the set of isotopy classes of essential simple closed curves on X . If $\gamma \in \mathcal{S}$, then its ρ -length is defined by

$$L_\rho(\gamma) = \inf_{\gamma'} \int_{\gamma'} \rho(z) |dz|,$$

where the infimum is taken over all simple closed curves γ' in the isotopy class of γ .

With the above notations, we may define the *extremal length* of γ on X by

$$\text{Ext}_\gamma(X) = \sup_\rho \frac{L_\rho^2(\gamma)}{\text{Area}_\rho(X)},$$

where $\rho(z)|dz|$ ranges over all conformal metrics on X such that $0 < \text{Area}_\rho(X) < \infty$.

An important property of $\text{Ext}_\gamma(X)$ is that it has several equivalent definitions. Note that the above definition is analytic. There is a geometric definition of $\text{Ext}_\gamma(X)$:

$$\text{Ext}_\gamma(X) = \inf_C \frac{1}{\text{mod}(C)},$$

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where the infimum is taken over all embedded cylinders C in X with core curves isotopic to γ and $\text{mod}(C)$ is the conformal modulus of C . As pointed out by Kerckhoff [10], in the estimation of extremal length, the analytic definition is useful for finding lower bounds, while the geometric definition is useful for finding upper bounds.

If we denote by $\Phi(\gamma)$ the Jenkins-Strebel differential whose vertical trajectories are isotopic to γ , then it was observed by Kerckhoff [10] that (see Proposition 2.1 below)

$$\text{Ext}_\gamma(X) = \int_X |\Phi(\gamma)|.$$

The aim of this paper is to study the variation of extremal length function on Teichmüller space. The first variational formula of $\text{Ext}_\gamma(\cdot)$ along any differential path on Teichmüller space was first given by Gardiner [6] (see also Remark 3.3 below). However, there is no explicit formula for higher-order variations.

In contrast with extremal length, variational formulas of hyperbolic length are thoroughly studied by Kerckhoff [11], Wolpert [22], [23], [24] and Wolf [21]. The convexity of hyperbolic length along earthquake paths or Weil-Petersson geodesics was used to solve the Nielsen realization problem [11] [23]. The estimates of Weil-Petersson gradient, Hessian and covariant derivative of the gradient of geodesic length are important in the study of Weil-Petersson curvature expansions and geodesic flow (see Wolpert [25]).

In this paper, we will identify $\text{Ext}_\gamma(X)$ with the energy of a harmonic map from X (to be more precise, the universal cover of X) to a real tree, another equivalent definition of $\text{Ext}_\gamma(X)$ which was discovered by Wolf [20]. Such a definition allows us to compute the second variations of $\text{Ext}_\gamma(\cdot)$ along Weil-Petersson and Teichmüller geodesics.

Kerckhoff [10] has discovered an elegant and useful way to compute the Teichmüller distance in terms of extremal length. It is an open question that whether Teichmüller geodesic balls are convex. Recently, Lenzhen and Rafi [12] proved that extremal length functions are quasi-convex along any Teichmüller geodesic. As a corollary, they proved the quasiconvexity of Teichmüller geodesic balls. We hope that our study of extremal length variations may be related to the convexity of Teichmüller metric.

1.1. Statement of main results. To state our results, we need to fix some terminology.

Let S be a smooth closed surface of negative Euler characteristic, and let $\mathcal{T}(S)$ be the Teichmüller space of (isotopy classes of) marked hyperbolic structures on S . For simplicity, for a Riemann surface X or a hyperbolic metric $g = g(z)|dz|^2$ on S , we will use the same notation X or (S, g) to represent its equivalent class in $\mathcal{T}(S)$. Note that the definition of extremal length only depends on the isotopy class of hyperbolic structures and the isotopy class of γ , thus for each fixed $\gamma \in \mathcal{S}$, $\text{Ext}_\gamma(\cdot)$ defines a (real analytic) function on $\mathcal{T}(S)$.

The Teichmüller space $\mathcal{T}(S)$ is a complex manifold endowed with a Kähler metric, the Weil-Petersson metric, which we will define immediately. Recall that tangent vectors to Teichmüller space at a point (S, g) are represented by

Beltrami differentials of the form $\mu = \frac{\bar{\Phi}}{g}$, where Φ is a holomorphic quadratic differential on (S, g) . The Weil-Petersson Riemannian inner product of two such tangent vectors is the L^2 inner product

$$\langle \frac{\bar{\Phi}}{g}, \frac{\bar{\Psi}}{g} \rangle = \operatorname{Re} \int_S \frac{\Phi}{g} \frac{\bar{\Psi}}{g} d\operatorname{Area}_g.$$

The Weil-Petersson metric is not complete, but it is geodesically convex [23]. This means that any two points in $\mathcal{T}(S)$ can be joined by a unique Weil-Petersson geodesic.

Our main result is:

Theorem 1.1. *Let $\Gamma_i(t), i = 1, 2$ be any two Weil-Petersson geodesics on $\mathcal{T}(S)$ with $\Gamma_1(0) = \Gamma_2(0) = X$ and $\frac{d}{dt}\Gamma_1(0) = \mu, \frac{d}{dt}\Gamma_2(0) = i\mu$, where i denotes the almost complex structure of $\mathcal{T}(S)$. Then*

$$\frac{d^2}{dt^2}|_{t=0}\operatorname{Ext}_\gamma(\Gamma_1(t)) + \frac{d^2}{dt^2}|_{t=0}\operatorname{Ext}_\gamma(\Gamma_2(t)) > 0.$$

The same result is also true for the Teichmüller metric.

The method of our computations relies on a result of Wolf [20] that the extremal length $\operatorname{Ext}_\gamma(X), X \in \mathcal{T}(S)$ is equal to half of the energy of a harmonic map from X to a \mathbb{R} -tree (see Section 3 for definition). It turns out that harmonic maps theory, which has been successfully applied by Wolf [18] in the study of Thurston compactification and Weil-Petersson geometry, may be useful to study extremal length functions.

Recall that a real-valued function on a complex manifold is *strictly pluri-subharmonic* if its Levi 2-form (see Section 5 for definition) is positive definite. Since the Weil-Petersson metric is Kähler, due to a result of Deligne, Griffiths, Morgan and Sullivan [3] (see also [15]), we have the following direct corollary.

Theorem 1.2. *Given any $\gamma \in \mathcal{S}$, the extremal length function $\operatorname{Ext}_\gamma(\cdot)$ is strictly pluri-subharmonic on $\mathcal{T}(S)$.*

In general, an upper semi-continuous function F on a complex manifold M is said to be *plurisubharmonic* if and only if for any holomorphic map $\phi : \Delta \rightarrow M$, the function $F \circ \phi : \Delta \rightarrow \mathbb{R}$ is subharmonic, where Δ denotes the unit disk. By density of weighted simple closed curves in the space of measured foliations, we have the following corollary.

Corollary 1.3. *Given any measured foliation \mathcal{F} on S , the extremal length function $\operatorname{Ext}_\mathcal{F}(\cdot)$ is pluri-subharmonic on $\mathcal{T}(S)$.*

Using a different method, Alexander Vasil'ev [16] showed the extremal length of some maximally rational measured foliations are locally harmonic in $\mathcal{T}(S)$.

We also obtain a new proof of the following well-known result.

Corollary 1.4. *Teichmüller space is a Stein manifold.*

Proof. Let $\{\gamma_i\}$ be a finite set of simple closed curves filling the surface S and set

$$L(\cdot) = \sum_{\gamma_i} \operatorname{Ext}_{\gamma_i}(\cdot).$$

By definition of extremal length, $L(\cdot)$ has a universal lower bound

$$(1) \quad \sum_{\gamma_i} \frac{\ell_{\gamma_i}^2(\cdot)}{2\pi|\chi(S)|},$$

where $\ell_{\gamma_i}(\cdot)$ denotes the hyperbolic length of γ_i . Since (1) is a proper function on $\mathcal{T}(S)$ (see Kerckhoff [11]), $L(\cdot)$ is also proper. It follows from Theorem 1.2 that $L(\cdot)$ defines a proper strictly pluri-subharmonic function on $\mathcal{T}(S)$. □

Remark 1.5. Note that Tromba [15] has proved that the Dirichlet energy function is pluri-subharmonic on $\mathcal{T}(S)$. More explicitly, fix a hyperbolic structure g_0 on S , for any $X \in \mathcal{T}(S)$, there exists a unique harmonic maps $h : X \rightarrow (S, g_0)$ homotopic to the identity map of S . Denote the energy of h by $E(X)$. It was shown by Tromba [15] that $E(\cdot)$ defines a pluri-subharmonic function on $\mathcal{T}(S)$ and the Hessian of $E(\cdot)$ is exactly the Weil-Petersson Riemannian metric. The fact that the target surface (S, g_0) has constant curvature -1 plays an important role in Tromba's proof. While in our discussion, the target is a \mathbb{R} -tree, which is not of constant curvature -1 . On the other hand, the convexity of Dirichlet energy function, with fixed domain and varied targets, is obtained by Wolf [18] and Yamada [26].

Note that a bounded non-positive strictly plurisubharmonic exhaustion function on $\mathcal{T}(S)$ defined in term of hyperbolic length functions was given by Yeung [27].

1.2. Organization of the paper. We recall Wolf's realization of extremal length as energy of harmonic maps in Section 2. In Section 3, we obtain a formal formula for the second variation of extremal length. Such a formula is simplified in Section 4, where we studies the variational vector field of harmonic maps. Section 5 is devoted to the proof of Theorem 1.2. Finally we discuss some relation between the extremal length variation and Teichmüller metric.

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2. PRELIMINARIES

In this section, we recall Wolf's treatment on realizing measured foliations via holomorphic quadratic differentials of harmonic maps from Riemann surfaces to \mathbb{R} -trees. It follows from his result that the extremal length $\text{Ext}_\gamma(X)$ can be realized as half of the energy of a harmonic map.

Let us first describe some necessary background materials about quadratic differential and measured foliation. For more details, one may refers to the book of Strebel [5] and Kerckhoff's thesis [10].

A *holomorphic quadratic differential* Φ on (S, g) is a $(2, 0)$ -tensor locally given by $\Phi = \Phi(z)dz^2$, where $\Phi(z)$ is holomorphic. Any holomorphic quadratic differential $\Phi = \Phi(z)dz^2$ determines a singular metric $|\Phi(z)||dz|^2$, with finitely many singular points corresponding to the zeros of Φ . The

total area of S in this metric is given by

$$\int_S |\Phi| = \int_S |\Phi(z)| |dz|^2.$$

Let $QD(g)$ be the space of holomorphic quadratic differentials on (S, g) .

A *measured foliation* on S is a foliation (with a finite number of singularities) endowed with an invariant transverse measure. The allowed singularities are topologically the same as those that occur at $z = 0$ in the line field $z^{p-2}dz^2$, $p \geq 3$. The *intersection number* $i(\gamma, \mathcal{F})$ of a simple closed curve γ with a measured foliation \mathcal{F} with transverse measure μ is defined by

$$\inf_{\gamma'} \int_{\gamma'} d\mu,$$

where the infimum is taken over all simple closed curves γ' in the isotopy class of γ . Two measured laminations \mathcal{F} and \mathcal{F}' are said to be *measured equivalent* if, for all simple closed curves γ , the geometric intersection numbers satisfy $i(\gamma, \mathcal{F}) = i(\gamma, \mathcal{F}')$. Denote by \mathcal{MF} the space of equivalence classes of measured foliations.

There is a special class of measured foliations that have the property that the complement of the critical leaves is homeomorphic to a cylinder. The leaves of the foliation on the cylinder are all freely homotopic to a simple closed curves γ . Such a foliation is completely determined as a point in \mathcal{MF} by the height a of the cylinder and the isotopy class of γ . Denote such a foliation by (γ, a) or $a\gamma$. Thurston showed that \mathcal{MF} is homeomorphic to a open ball of dimension $6g - g$ and there is an embedding $\mathcal{S} \times \mathbb{R}_+ \mapsto \mathcal{MF}$ whose image is dense [5].

The definition of extremal length can be extended from simple closed curves to measured foliations. There is a unique continuous extension of the extremal length function from \mathcal{S} to \mathcal{MF} such that $\text{Ext}_{a\gamma}(X) = a^2 \text{Ext}_\gamma(X)$ (see Kerckhoff [10]).

An element $\Phi \in QD(g)$ gives rise to a pair of transverse measured foliations $\mathcal{F}_h(\Phi)$ and $\mathcal{F}_v(\Phi)$ on S , called the horizontal foliation and vertical foliation of Φ respectively. The leaves of these foliations are given by setting the imaginary part (resp. real part) of Ψ equal to a constant. In a neighborhood of a nonsingular point, there are natural coordinates $z = x + iy$ so that the leaves of \mathcal{F}_h are given by $y = \text{constant}$, and the transverse measure of \mathcal{F}_h is $|dy|$. The leaves of \mathcal{F}_v are given by $x = \text{constant}$, and the transverse measure is $|dx|$. The foliations $\mathcal{F}_h(\Phi)$ and $\mathcal{F}_v(\Phi)$ have zero set of q as their common singular set, and at each zero of order k they have a $k + 2$ -pronged singularity, locally modeled on the singularity at the origin of $z^k dz^2$.

According to a fundamental theorem of Hubbard and Masur [8], if \mathcal{F} is a measured foliation on (S, g) , then there is a unique holomorphic quadratic differential $\Phi(\mathcal{F}) \in QD(g)$ such that \mathcal{F} is measured equivalent to $\mathcal{F}_v(\Phi(\mathcal{F}))$.

The following fact is observed by Kerckhoff [10].

Proposition 2.1. *Fixed a hyperbolic metric g on S . The extremal length of any measured foliation \mathcal{F} is equal to the area of $\Phi(\mathcal{F})$, that is,*

$$\text{Ext}_{\mathcal{F}}(g) = \int_S |\Phi(\mathcal{F})|.$$

Proof. The proof we give here can be found in Ivanov [9]. We include it for the sake of completeness.

By continuity and the density of weighted simple closed curve in \mathcal{MF} , it suffices to prove the proposition for the case that $\mu = a\gamma \in \mathcal{MF}$, where γ is (the isotopy class of) a simple closed curve and $a > 0$.

Let Φ be the one-cylinder Strebel differential on X determined by $a\gamma$. The complement of the vertical critical leaves of Φ is a cylinder foliated by circles isotopic to γ . Let us also set $\rho = |\Phi|^{1/2}|dz|$. Then ρ is a flat metric on S , with a finite number of singular points, which are conical singularities. Measured in the flat metric ρ , the circumference and height of the cylinder are equal to $L_\rho(\gamma)$ and a , respectively. By a theorem of Jenkins-Strebel [14], the extremal length $\text{Ext}_\gamma(g)$ of γ is equal to

$$\text{Ext}_\gamma(g) = \frac{L_\rho(\gamma)}{a},$$

where $\rho = |\Phi|^{1/2}|dz|$.

Since the area $\text{Area}(\rho) = \int_S |\Phi(z)||dz|^2$ of the cylinder is equal to $aL_\rho(\gamma)$, we have

$$\text{Ext}_{a\gamma}(g) = a^2 \text{Ext}_\gamma(g) = aL_\rho(\gamma) = \text{Area}(\rho).$$

□

Consider a measured foliation (\mathcal{F}, μ) on (S, g) . Lift this measured foliation to a measured foliation $(\tilde{\mathcal{F}}, \tilde{\mu})$ on the universal cover (\tilde{S}, g) . Let T be the leaf space of $\tilde{\mathcal{F}}$. There is a natural projection $\pi : \tilde{S} \rightarrow T$. We can define a metric d on T by pushing forward the measure $\tilde{\mu}$ by the projection π , a.e., $d = \pi_* \tilde{\mu}$. In this way, (T, d) becomes a real tree (and a real line whenever \mathcal{F} is a simple closed curve). The fundamental group $\pi_1(S)$ acts by isometries on (T, d) and the map π is equivalent with respect to this action.

With the above terminology, now we may state the following result of Wolf (see Proposition 3.1, [20]) in the form we need in this paper.

Proposition 2.2. *There is a $\pi_1(S)$ -equivariant map $\omega : (\tilde{S}, g) \rightarrow (T, d)$ which is equivariantly homotopic to $\pi : (\tilde{S}, g) \rightarrow (T, d)$. Off a discrete set, ω is locally a harmonic projection to a Euclidean line.*

Moreover, the vertical measured foliation of the Hopf differential $\Phi = \langle \omega_z, \omega_z \rangle_d dz^2$ of ω is measured equivalent to (\mathcal{F}, μ) .

In this paper, the map ω in Proposition 2.2 is called a (equivariant) harmonic map from (S, g) to (T, d) (the R-tree associated with (\mathcal{F}, μ)). The energy of ω is given by

$$\begin{aligned} E(\omega, g, d) &= \frac{1}{2} \int_S |\omega_z|_d^2 + |\omega_{\bar{z}}|_d^2 dz d\bar{z} \\ &= \int_S |\omega_z|_d^2 dz d\bar{z} \\ &= \int_S |\Phi| \\ &= \text{Ext}_{\mathcal{F}}(g). \end{aligned}$$

In the above equations, the integral domain S is considered as a fundamental domain of $\pi_1(S)$ on the universal cover \tilde{S} . Since the harmonic map is $\pi_1(S)$ -equivariant, the energy is well-defined. The second equality holds since the Jacobi $|\omega_z|^2 - |\omega_{\bar{z}}|^2 = 0$ almost every where. The last equality holds since the Hopf differential Φ of ω is equal to $\Phi(\mathcal{F})$ and the area of $\Phi(\mathcal{F})$ is equal to $\text{Ext}_{\mathcal{F}}(g)$ (by Proposition 2.1). As a result, in our language, we have the following result:

Proposition 2.3. *The extremal length $\text{Ext}_{\gamma}(g)$ is realized as the energy of the harmonic map ω .*

Note that the extremal length function $\text{Ext}_{\gamma}(\cdot)$ of a simple closed curve γ is a real analytic function on Teichmüller space, since it corresponds to the energy of harmonic maps from Riemann surfaces to an interval, with the required analyticity coming from Eells-Lemaire [4]. For a general measured foliation \mathcal{F} , Wentworth [17] showed that $\text{Ext}_{\mathcal{F}}(\cdot)$ is differential on $\mathcal{T}(S)$.

3. SECOND VARIATION OF EXTREMAL LENGTH

Let us first fix some notations. Let (S, g_t) be a differential family of hyperbolic metrics, and let $z^t : (S, g_0|dz|^2) \rightarrow (S, g_t|dz_t|^2)$ be the corresponding family of quasiconformal maps with (corresponding) Beltrami differential $\mu(t) = t\mu + o(t)$, where z_t is the local conformal coordinates of (S, g_t) and z is the local conformal coordinates of (S, g_0) . We may consider (S, g_t) as a path in $\mathcal{T}(S)$, with tangent vector μ at the basis point (S, g_0) . Moreover, we always assume that μ is a harmonic Beltrami differential $\frac{\Phi}{g_0}$. Since we mainly consider the variation along Teichmüller geodesics or Weil-Petersson geodesics, we can also assume that $\ddot{\mu} = \frac{d^2}{dt^2}\mu(t)|_{t=0} \equiv 0$.

In this section, we will compute the second variation of the extremal length function of a fixed simple closed curve γ . One may wish to extend the computations to the extremal length function of any measured foliation \mathcal{F} . However, we don't know the regularity of the function $\text{Ext}_{\mathcal{F}}(\cdot)$ in general. Another possible generalization is to study the variation of $\text{Ext}_{\mathcal{F}_t}(g_t)$ where \mathcal{F}_t is varied (such a case appears when we study the Teichmüller distance function between a Teichmüller geodesic and a basis point).

Let (T, d) be the R-tree associated with γ . Let $\omega^t : (S, g_t) \rightarrow (T, d)$ be the corresponding harmonic map whose energy is equal to $2\text{Ext}_{\gamma}(g_t)$. Denote by $E(\omega^t, g_t)$ the energy of the corresponding harmonic map, which is equal to the extremal length function. That is,

$$E(\omega^t, g_t) = \int_S \left| \frac{\partial \omega^t}{\partial z_t} \right|^2 dz_t d\bar{z}_t.$$

We assume that $\omega^t = \omega + t\dot{\omega} + o(t)$, where $\dot{\omega}$, defined on (S, g_0) , is the *variational vector field of ω^t at $t = 0$* .

Remark 3.1. We have pointed out that, in the case that γ is a simple closed curve, the R-tree (T, d) associated with γ is isometric to the real line. In this case, the uniqueness of ω^t up to parallel translation in T is due to Hartman [7]. In particular, the variational vector field $\dot{\omega}$ is well-defined.

First we separate the overall variation into a term that refers only the second variation of the metrics g_t and a term that refers only to the second variation of the maps ω^t . Such a separation is quite standard for a variational functional (see Wolf [21]).

Since the energy of a harmonic map is stationary with respect to variations of the map,

$$D_1 E(\omega^t, g_t)[\dot{\omega}] = 0$$

and then

$$\begin{aligned} 0 &= \frac{d}{dt} D_1 E(\omega^t, g_t)[\dot{\omega}] \\ &= D_{11}^2 E(\omega^t, g_t)[\dot{\omega}, \dot{\omega}] + D_{12}^2 E(\omega^t, g_t)[\dot{\omega}, \dot{g}]. \end{aligned}$$

Thus

$$D_{11}^2 E(\omega^t, g_t)[\dot{\omega}, \dot{\omega}] = -D_{12}^2 E(\omega^t, g_t)[\dot{\omega}, \dot{g}].$$

The second variation of extremal length function is given by

$$\begin{aligned} \frac{d^2}{dt^2} E(\omega^t, g_t) &= D_{11}^2 E(\omega_0, g_0)[\dot{\omega}, \dot{\omega}] + 2D_{12}^2 E(\omega_0, g_0)[\dot{\omega}, \dot{g}] \\ &\quad + D_{22}^2 E(\omega_0, g_0)[\dot{g}, \dot{g}]. \end{aligned}$$

Thus, we have

$$(2) \quad \frac{d^2}{dt^2} E(\omega^t, g_t) = -D_{11}^2 E(\omega^t, g_t)[\dot{\omega}, \dot{\omega}] + D_{22}^2 E(\omega^t, g_t)[\dot{g}, \dot{g}].$$

Remark 3.2. One may ask the question that whether $D_{22}^2 E(\omega^t, g_t)[\dot{g}, \dot{g}]$ is larger than $D_{11}^2 E(\omega^t, g_t)[\dot{\omega}, \dot{\omega}]$. An affirmative answer will show that the extremal length function is convex.

For simplicity, we denote $\omega^0 = \omega$. As above, we have

$$E(\omega^t, g_t) = \int_S \left| \frac{\partial \omega^t}{\partial z_t} \right|^2 dz_t d\bar{z}_t.$$

Note that

$$\begin{aligned} dz_t &= (z_t)_z dz + (z_t)_{\bar{z}} d\bar{z}, \\ d\bar{z}_t &= (\bar{z}_t)_z dz + (\bar{z}_t)_{\bar{z}} d\bar{z}, \\ \mu(t) &= \frac{(z_t)_{\bar{z}}}{(z_t)_z}, \end{aligned}$$

$$(3) \quad dz_t d\bar{z}_t = |(z_t)_z|^2 (1 - |\mu(t)|^2) dz d\bar{z}.$$

It follows from the chain rule of differential that

$$\begin{aligned} \omega_{z_t} &= \frac{\omega_z (\bar{z}_t)_{\bar{z}} - \omega_{\bar{z}} (\bar{z}_t)_z}{|(z_t)_z|^2 (1 - |\mu(t)|^2)} \\ &= \frac{\omega_z - \overline{\mu(t)} \omega_{\bar{z}}}{(z_t)_z (1 - |\mu(t)|^2)}. \end{aligned}$$

As a result,

$$\begin{aligned} (4) \quad |\omega_{z_t}|^2 &= \frac{(\omega_z - \overline{\mu(t)} \omega_{\bar{z}})(\omega_z - \overline{\mu(t)} \omega_{\bar{z}})}{|(z_t)_z|^2 (1 - |\mu(t)|^2)^2} \\ &= \frac{|\omega_z|^2 + |\mu(t)|^2 |\omega_{\bar{z}}|^2 - 2\operatorname{Re}(\mu(t) \omega_z \bar{\omega}_{\bar{z}})}{|(z_t)_z|^2 (1 - |\mu(t)|^2)^2}. \end{aligned}$$

Combining equation (3) with (4), we have

$$E(\omega, g_t) = \int_S \frac{|\omega_z|^2 + |\mu(t)|^2 |\omega_{\bar{z}}|^2 - 2\operatorname{Re}(\mu(t)\omega_z \bar{\omega}_z)}{1 - |\mu(t)|^2} dz d\bar{z}.$$

Assume that $\mu(t) = t\mu + o(t^2)$, that is, $\dot{\mu} = \mu$ and $\ddot{\mu} \equiv 0$. Then we have

$$E(\omega, g_t) = \int_S \frac{|\omega_z|^2 + t^2 |\mu|^2 |\omega_{\bar{z}}|^2 - t 2\operatorname{Re}(\mu \omega_z \bar{\omega}_z)}{1 - t^2 |\mu|^2} dz d\bar{z} + o(t^2).$$

Now we consider the variation of $E(\omega, g_t)$. Note that

$$\begin{aligned} \frac{d}{dt} E(\omega, g_t) &= \int_S \frac{2t |\mu|^2 |\omega_{\bar{z}}|^2 - 2\operatorname{Re}(\mu \omega_z \bar{\omega}_z)}{1 - t^2 |\mu|^2} dz d\bar{z} \\ &\quad - \int_S (|\omega_z|^2 + t^2 |\mu|^2 |\omega_{\bar{z}}|^2 - t 2\operatorname{Re}(\mu \omega_z \bar{\omega}_z)) \frac{-2t |\mu|^2}{|1 - t^2 |\mu|^2|^2} dz d\bar{z} + o(t). \end{aligned}$$

It follows that

$$(5) \quad \frac{d^2}{dt^2} \Big|_{t=0} E(\omega, g_t) = 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z}.$$

Remark 3.3. Note that by letting $t = 0$, it follows from the above computation of the first variation that

$$\frac{d}{dt} \Big|_{t=0} E(\omega, g_t) = -2\operatorname{Re} \int_S \mu \omega_z \bar{\omega}_z dz d\bar{z}.$$

Since $\frac{d}{dt} \Big|_{t=0} E(\omega^t, g_t) = D_1 E(\omega_0, g_0)[\dot{\omega}] + D_2 E(\omega_0, g_0)[\dot{g}]$ and $D_1 E(\omega_0, g_0)[\dot{\omega}] = 0$, we get the following first variation formula:

$$(6) \quad \frac{d}{dt} \Big|_{t=0} E(\omega^t, g_t) = -2\operatorname{Re} \int_S \Phi \mu$$

where $\Phi = \omega_z \bar{\omega}_z dz^2 \in QD(g_0)$, which is the Hubbard-Masur differential for γ at (S, g_0) . Actually, the above formula is valid for any measured foliation (see Gardiner [6] and Wentworth [17]).

Our next step is to evaluate the term $D_{22}^2 E(\omega_0, g_0)[\dot{\omega}, \dot{\omega}] = \frac{d^2}{dt^2} \Big|_{t=0} E(\omega^t, g_0)$ in (2).

Set $\omega^t = \omega + t\dot{\omega} + \frac{t^2}{2}\ddot{\omega} + o(t^2)$. Then

$$E(\omega^t, g_0) = \int_S |\omega_z^t|^2 dz d\bar{z}$$

and

$$\begin{aligned} \left| \frac{\partial \omega^t}{\partial z} \right|^2 &= (\omega_z + t\dot{\omega}_z + \frac{t^2}{2}\ddot{\omega}_z + o(t^2)) \overline{(\omega_z + t\dot{\omega}_z + \frac{t^2}{2}\ddot{\omega}_z + o(t^2))} \\ &= |\omega_z|^2 + t 2\operatorname{Re}(\dot{\omega}_z \bar{\omega}_z) + t^2 (|\dot{\omega}_z|^2 + \operatorname{Re}(\omega \ddot{\omega}_{z\bar{z}})) + o(t^2). \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} E(\omega^t, g_0) &= 2\operatorname{Re} \int_S \dot{\omega}_z \bar{\omega}_z dz d\bar{z}, \text{ integration by parts} \\ &= -2\operatorname{Re} \int_S \dot{\omega} \bar{\omega}_{z\bar{z}} dz d\bar{z}, \text{ by harmonicity} \\ &= 0. \end{aligned}$$

And then

$$\begin{aligned}
(7) \quad \frac{d^2}{dt^2}|_{t=0}E(\omega^t, g_0) &= 2 \int_S |\dot{\omega}_z|^2 dz d\bar{z} + 2 \operatorname{Re} \int_S \omega_z \bar{\omega}_{\bar{z}} dz d\bar{z} \\
&= 2 \int_S |\dot{\omega}_z|^2 dz d\bar{z} - 2 \operatorname{Re} \int_S \omega_{z\bar{z}} \bar{\omega} dz d\bar{z} \\
&= 2 \int_S |\dot{\omega}_z|^2 dz d\bar{z}.
\end{aligned}$$

Combining (2), (5) and (7), we have

$$\frac{d^2}{dt^2}|_{t=0}E(\omega^t, g^t) = 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} - 2 \int_S |\dot{\omega}_z|^2 dz d\bar{z}.$$

In conclusion, we have the following:

Theorem 3.4. *For any simple closed curve γ , if g_t is a differential family of hyperbolic structures in $\mathcal{T}(S)$ with Beltrami differential $\mu(t) = t\mu + o(t^2)$ at $t = 0$, and ω is the harmonic map from (\tilde{S}, g_0) to the \mathbb{R} -tree determined by γ , then*

$$\begin{aligned}
\frac{d^2}{dt^2}|_{t=0}Ext_\gamma(g_t) &= 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} - 2 \int_S |\dot{\omega}_z|^2 dz d\bar{z} \\
&= 4 \int_S |\mu|^2 |\omega_z|^2 dz d\bar{z} - 2 \int_S |\dot{\omega}_z|^2 dz d\bar{z}.
\end{aligned}$$

4. THE VECTOR FIELD $\dot{\omega}$

We have shown that

$$\frac{d^2}{dt^2}|_{t=0}E(\omega^t, g_0) = 2 \int_S |\dot{\omega}_z|^2 dz d\bar{z}.$$

In order to compare this formula to (5), it is important to find an expression for $\dot{\omega}$ or $\dot{\omega}_z$ in terms of μ and ω . Note that since ω is a harmonic map from (S, g_0) to a \mathbb{R} -tree (in fact, an interval), we can take ω to be real and then (5) becomes

$$(8) \quad \frac{d^2}{dt^2}|_{t=0}E(\omega, g_t) = 4 \int_S |\mu|^2 |\Phi| dz d\bar{z},$$

where $\Phi dz d\bar{z} = \omega_z \bar{\omega}_{\bar{z}} dz d\bar{z}$ is the Hopf differential of ω . So it will be interesting if one can give an expression of $\dot{\omega}$ or $\dot{\omega}_z$ in term of μ and Φ .

Consider the harmonic map equation $\frac{\partial^2 \omega^t}{\partial z_t \partial \bar{z}_t} = 0$, which may be denoted by $H(\omega^t, z_t) = 0$. Differentiating in t , we find that

$$(9) \quad -\frac{d}{dt}|_{t=0}H(\omega, z_t) = \frac{d}{dt}|_{t=0}H(\omega^t, z) = \dot{\omega}_{z\bar{z}}.$$

To compute $\frac{d}{dt}|_{t=0}H(\omega, z_t)$, we express the operator $\frac{\partial^2}{\partial z_t \partial \bar{z}_t}$ as

$$\left\{ \frac{1}{1 - t^2 |\dot{\mu}|^2} \frac{1}{(z_t)_z} (\partial_z - t \bar{\mu} \partial_{\bar{z}}) \right\} \left\{ \frac{1}{1 - t^2 |\dot{\mu}|^2} \frac{1}{(z_t)_{\bar{z}}} (-t \mu \partial_z + \partial_{\bar{z}}) \right\} + o(t)$$

$$= \left\{ \frac{1}{(z_t)_z} (\partial_z - t\bar{\mu}\partial_{\bar{z}}) \frac{1}{(z_t)_z} (-t\dot{\mu}\partial_z + \partial_{\bar{z}}) + o(t) \right\}.$$

As a result,

$$\begin{aligned} -\frac{d}{dt}|_{t=0} H(\omega, z_t) &= \bar{\mu}\omega_{\bar{z}\bar{z}} + \dot{z}_{\bar{z}}\bar{\omega}_{\bar{z}} + \dot{z}_z\omega_{z\bar{z}} + \dot{\mu}_z\omega_z - \dot{\mu}\omega_{zz} \\ &= \dot{\mu}\omega_{zz} + \bar{\mu}\omega_{\bar{z}\bar{z}} + \dot{z}_{\bar{z}}\bar{\omega}_{\bar{z}} + \dot{\mu}_z\omega_z \end{aligned}$$

The last equality holds since $\omega_{z\bar{z}} = 0$.

Ahlfors [1] proved that $\mu = \dot{z}_{\bar{z}}$ (in the sense of distribution), hence $\overline{\dot{z}_{\bar{z}}} = \overline{\mu}_z$, and then

$$\begin{aligned} -\frac{d}{dt}|_{t=0} H(\omega, z_t) &= \dot{\mu}\omega_{zz} + \bar{\mu}\omega_{\bar{z}\bar{z}} + \overline{\mu}_z + \dot{\mu}_z\omega_z \\ (10) \qquad \qquad \qquad &= \frac{\partial}{\partial z}(\mu\omega_z) + \frac{\partial}{\partial \bar{z}}(\bar{\mu}\omega_{\bar{z}}). \end{aligned}$$

It follows from (9) and (10) that

$$(11) \qquad \qquad \qquad \dot{\omega}_{z\bar{z}} = \frac{\partial}{\partial z}(\mu\omega_z) + \frac{\partial}{\partial \bar{z}}(\bar{\mu}\omega_{\bar{z}}).$$

Using (11) and integration by part, we have

$$\begin{aligned} \int_S |\dot{\omega}_z|^2 dz d\bar{z} &= - \int_S \bar{\omega} \dot{\omega}_{z\bar{z}} dz d\bar{z} \\ &= - \int_S \frac{\partial}{\partial z}(\mu\omega_z) \bar{\omega} + \frac{\partial}{\partial \bar{z}}(\bar{\mu}\omega_{\bar{z}}) \bar{\omega} dz d\bar{z} \\ (12) \qquad \qquad \qquad &= \int_S \mu\omega_z \bar{\omega}_z + \bar{\mu}\omega_{\bar{z}} \bar{\omega}_{\bar{z}} dz d\bar{z}. \end{aligned}$$

The inequality (12) will be used in the proof of Theorem 1.2.

5. PLURI-SUBHARMONICITY

In this section we prove that the extremal length function is strictly pluri-subharmonic on Teichmüller space.

Recall that a real valued function F on a complex manifold M is (*strictly*) *pluri-subharmonic* if the *Levi-form* $\partial\bar{\partial}F$ is (strictly) pluri-subharmonic at each point of M . Recall that $\partial\bar{\partial}F$ is a 2-form defined by

$$\partial\bar{\partial}F = \frac{\partial^2 F}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha d\bar{z}^\beta$$

in holomorphic coordinates.

If $\xi = \{\xi^\alpha\}$ and $\eta = \{\eta^\alpha\}$ are tangent vectors to M at a point z , then the value of this 2-form on tangent vectors is

$$\frac{\partial^2 F(z)}{\partial z^\alpha \partial \bar{z}^\beta} \xi^\alpha \bar{\eta}^\beta.$$

Since on a complex manifold the transition maps are holomorphic, the sign of $\frac{\partial^2 F(z)}{\partial z^\alpha \partial \bar{z}^\beta} \xi^\alpha \bar{\eta}^\beta$ is independent of the choice of the holomorphic coordinates.

As a consequence, if $\frac{\partial^2 F(z)}{\partial z^\alpha \partial \bar{z}^\beta} \xi^\alpha \bar{\eta}^\beta \geq 0$ for any $\xi = \{\xi^\alpha\}$ and $\eta = \{\eta^\alpha\}$, we say that F is pluri-subharmonic at z and strictly pluri-subharmonic if $\frac{\partial^2 F(z)}{\partial z^\alpha \partial \bar{z}^\beta} \xi^\alpha \bar{\eta}^\beta > 0$.

The notion of pluri-subharmonic functions is the natural generalization of the notion of subharmonic functions of single complex variable, and it is related to several important notions in several complex variable theory, such as domain of holomorphy and so on.

In order to compute the Levi-form of the extremal length function $E(\cdot) = \text{Ext}_\gamma(\cdot)$, we consider the Weil-Petersson metric on $\mathcal{T}(S)$. Since it is Kähler, the computation of the Levi-form is reduced to

$$\frac{d^2}{dt^2}|_{t=0}E(\mu_1(t)) + \frac{d^2}{dt^2}|_{t=0}E(\mu_2(t))$$

where $\mu_i(t), i = 1, 2$ are two Weil-Petersson geodesics, such that $\mu_1(0) = \mu_2(0) = \mu$ and $\frac{d}{dt}\mu_1(0) = \mu, \frac{d}{dt}\mu_2(0) = i\mu$. Note that here we have identified the family of Beltrami differentials $\mu_i(t)$ as a path in $\mathcal{T}(S)$, which is corresponding to a family of hyperbolic metrics $(S, g_t|dz_t|^2)$, satisfying $\frac{\partial z_t}{\partial \bar{z}} = \mu_i(t)\frac{\partial z_t}{\partial z}$.

5.1. Proof of pluri-subharmonicity. Let $\mu_i(t), i = 1, 2$ be two Weil-Petersson geodesics, such that $\mu_1(0) = \mu_2(0) = \mu$ and $\frac{d}{dt}\mu_1(0) = \mu, \frac{d}{dt}\mu_2(0) = i\mu$. In fact, we can assume that $\mu = \frac{\Phi}{g_0}$ and $\mu_1(t) = t\mu, \mu_2(t) = it\mu$, because $\mu(t) = t\frac{\Phi}{g_0}$ is Weil-Petersson geodesic at $t = 0$ (see Ahlfors [1]). We denote $\dot{\omega}(\mu)$ and $\dot{\omega}(i\mu)$ be the corresponding variation fields of harmonic maps.

By our discussion in Section 3, we have

$$(13) \quad \begin{cases} \frac{d^2}{dt^2}E(\mu_1(t)) = -D_{XX}^2E[\dot{\omega}(\mu), \dot{\omega}(\mu)] + D_{YY}E[\mu, \mu]. \\ \frac{d^2}{dt^2}E(\mu_2(t)) = -D_{XX}^2E[\dot{\omega}(i\mu), \dot{\omega}(i\mu)] + D_{YY}E[i\mu, i\mu]. \end{cases}$$

As we have shown,

$$\begin{cases} D_{YY}^2E[\mu, \mu] = 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} \\ D_{XX}^2E[\dot{\omega}(\mu), \dot{\omega}(\mu)] = 2 \int_S |\dot{\omega}_z(\mu)|^2 dz d\bar{z} \end{cases}$$

We have

$$(14) \quad \begin{aligned} & \frac{d^2}{dt^2}|_{t=0}E(\mu_1(t)) + \frac{d^2}{dt^2}|_{t=0}E(\mu_2(t)) \\ &= 4 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} \\ & \quad - 2 \int_S |\dot{\omega}_z(\mu)|^2 dz d\bar{z} - 2 \int_S |\dot{\omega}_z(i\mu)|^2 dz d\bar{z}. \end{aligned}$$

By equation (12), we have

$$\begin{aligned} \int_S |\dot{\omega}_z(\mu)|^2 dz d\bar{z} &= \int_S \bar{\dot{\omega}}_z(\mu) \mu \omega_z + \int_S \overline{\dot{\omega}_z(\mu)} \bar{\mu} \omega_{\bar{z}} dz d\bar{z}, \\ \int_S |\dot{\omega}_z(i\mu)|^2 dz d\bar{z} &= \int_S \bar{\dot{\omega}}_z(i\mu) i\mu \omega_z + \int_S \overline{\dot{\omega}_z(i\mu)} i\bar{\mu} \omega_{\bar{z}} dz d\bar{z} \\ &= \int_S i\bar{\dot{\omega}}_z(i\mu) \mu \omega_z + \int_S -i\overline{\dot{\omega}_z(i\mu)} \bar{\mu} \omega_{\bar{z}} dz d\bar{z}. \end{aligned}$$

As ω to be real, we have

$$\begin{aligned}
 & 2 \int_S |\dot{\omega}_z(\mu)|^2 dz d\bar{z} + 2 \int_S |\dot{\omega}_z(i\mu)|^2 dz d\bar{z} \\
 = & 2 \int_S \mu \omega_z [\bar{\omega}_z(\mu) + i \bar{\omega}_z(i\mu)] dz d\bar{z} + 2 \int_S \bar{\mu} \omega_{\bar{z}} [\bar{\omega}_z(\mu) - i \bar{\omega}_z(i\mu)] dz d\bar{z} \\
 \leq & 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} \\
 + & \frac{1}{2} \int_S |\bar{\omega}_z(\mu) + i \bar{\omega}_z(i\mu)|^2 dz d\bar{z} + \frac{1}{2} \int_S |\bar{\omega}_z(\mu) - i \bar{\omega}_z(i\mu)|^2 dz d\bar{z} \\
 = & 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} + \int_S |\dot{\omega}_z(\mu)|^2 dz d\bar{z} + \int_S |\dot{\omega}_z(i\mu)|^2 dz d\bar{z} \\
 + & \int_S \langle i \bar{\omega}_z(\mu), \dot{\omega}_z(i\mu) \rangle dz d\bar{z} - \int_S \langle i \bar{\omega}_z(\mu), \dot{\omega}_{\bar{z}}(i\mu) \rangle dz d\bar{z}.
 \end{aligned}$$

In the above computation, we have applied the following inequality:

$$2 \int_S |fg| dz d\bar{z} \leq 2 \int_S |f|^2 dz d\bar{z} + \frac{1}{2} \int_S |g|^2 dz d\bar{z}.$$

It follows from integration by part that the last two terms of the above inequality are equal to zero. Therefore we have proved that

$$\begin{aligned}
 & \int_S |\dot{\omega}_z(\mu)|^2 dz d\bar{z} + \int_S |\dot{\omega}_z(i\mu)|^2 dz d\bar{z} \\
 & \leq 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z},
 \end{aligned}$$

It follows that

$$(15) \quad \frac{d^2}{dt^2} \big|_{t=0} E(\mu_1(t)) + \frac{d^2}{dt^2} \big|_{t=0} E(\mu_2(t)) \geq 0.$$

As a result, we have shown that the extremal length function is pluri-subharmonic.

We have the following observation:

Proposition 5.1. *Then equation (14) can be wrote as*

$$(16) \quad \frac{d^2}{dt^2} \big|_{t=0} E(\mu_1(t)) + \frac{d^2}{dt^2} \big|_{t=0} E(\mu_2(t)) = 8 \int_S |\mu|^2 |\omega_z|^2 - 8 \int_S |\mu_{\bar{z}}|^2 |\omega_z|^2.$$

Proof. We know that ω is real. Write down the equation (11) as

$$\dot{\omega}_{z\bar{z}}(\mu) = 2 \operatorname{Re} \frac{\partial}{\partial z} (\mu \omega_z).$$

As a result, we see that

$$\dot{\omega}_{z\bar{z}}(i\mu) = 2 \operatorname{Re} \frac{\partial}{\partial z} (i\mu \omega_z) = -2 \operatorname{Im} \frac{\partial}{\partial z} (\mu \omega_z).$$

Then

$$|\dot{\omega}_z(\mu)|^2 dz d\bar{z} + |\dot{\omega}_z(i\mu)|^2 dz d\bar{z} = 4 \left| \frac{\partial}{\partial z} (\mu \omega_z) \right|^2.$$

$$\begin{aligned}
\int_S |\dot{\omega}_z(\mu)|^2 dz \bar{z} + |\dot{\omega}_z(i\mu)|^2 &= 4 \int_S \left| \frac{\partial}{\partial z} (\mu \omega_z) \right|^2 \\
&= -4 \int_S (\mu \omega_z)_{z\bar{z}} \overline{\mu \omega_z}, \text{ integration by part} \\
&= -4 \int_S (\mu_{\bar{z}} \omega_z)_z \overline{\mu \omega_z}, \text{ since } \omega_{z\bar{z}} = 0 \\
&= 4 \int_S \mu_{\bar{z}} \omega_z (\overline{\mu \omega_z})_z \\
&= 4 \int_S \mu_{\bar{z}} \omega_z (\overline{\mu_{\bar{z}} \omega_z}), \text{ since } \omega_{z\bar{z}} = 0 \text{ again} \\
&= 4 \int_S |\mu_{\bar{z}}|^2 |\omega_z|^2
\end{aligned}$$

Thus equation (14) is equal to

$$(17) \quad \frac{d^2}{dt^2} |_{t=0} E(\mu_1(t)) + \frac{d^2}{dt^2} |_{t=0} E(\mu_2(t)) = 8 \int_S |\mu|^2 |\omega_z|^2 - 8 \int_S |\mu_{\bar{z}}|^2 |\omega_z|^2.$$

□

5.2. Proof of strictly pluri-subharmonicity. Next we show that the left hand side of (15) is actually positive.

Since the image of ω is a \mathbb{R} -tree, the Jacobian $|\omega_z|^2 - |\omega_{\bar{z}}|^2 = 0$. Moreover we can take ω to be real and then $\omega_{\bar{z}} = \overline{\omega_z}$, $\dot{\omega}_{\bar{z}} = \overline{\dot{\omega}_z}$.

By Schwarz inequality, as we did before,

$$\begin{aligned}
&2 \int_S |\dot{\omega}_z(\mu)|^2 dz \bar{z} + 2 \int_S |\dot{\omega}_z(i\mu)|^2 dz \bar{z} \\
&= 2 \int_S \mu \omega_z (\overline{\dot{\omega}_z}(\mu) + i \overline{\dot{\omega}_z}(i\mu)) dz d\bar{z} + 2 \int_S \bar{\mu} \omega_{\bar{z}} (\overline{\dot{\omega}_z}(\mu) - i \overline{\dot{\omega}_z}(i\mu)) dz d\bar{z} \\
&\leq 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} \\
&\quad + \frac{1}{2} \int_S |\overline{\dot{\omega}_z}(\mu) + i \overline{\dot{\omega}_z}(i\mu)|^2 dz d\bar{z} + \frac{1}{2} \int_S |\dot{\omega}_z(\mu) - i \dot{\omega}_z(i\mu)|^2 dz d\bar{z} \\
&= 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} + \int_S |\dot{\omega}_z(\mu)|^2 dz \bar{z} + \int_S |\dot{\omega}_z(i\mu)|^2 dz d\bar{z} \\
&\quad + \int_S \langle i \overline{\dot{\omega}_z}(\mu), \dot{\omega}_z(i\mu) \rangle dz d\bar{z} - \int_S \langle i \overline{\dot{\omega}_z}(\mu), \dot{\omega}_{\bar{z}}(i\mu) \rangle dz d\bar{z}. (*)
\end{aligned}$$

If $\omega_z = 0$ at a point, then $\omega_{\bar{z}} = 0$ at this point too. Let $S_0 = \{z \in S : \omega_z = 0\}$. Then the integral

$$\int_{S_0} |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} = 0,$$

$$\int_{S_0} |\dot{\omega}_z(\mu)|^2 dz d\bar{z} = \int_{S_0} \overline{\dot{\omega}_z} \mu \omega_z + \int_{S_0} \overline{\dot{\omega}_z} \bar{\mu} \omega_{\bar{z}} dz d\bar{z} = 0,$$

and

$$\int_{S_0} |\dot{\omega}_z(i\mu)|^2 dz d\bar{z} = 0.$$

As a result, the points in S_0 have no contribution to the Schwarz inequality (*) and the inequality

$$\int_S |\dot{\omega}_z(\mu)|^2 dz \bar{z} + \int_S |\dot{\omega}_z(i\mu)|^2 dz \bar{z} \leq 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z}.$$

Therefore, we only need to consider the Schwarz inequality (*) in any neighborhood $\mathcal{U} \subset S$ where $\omega_z \neq 0$.

Suppose that $\omega_z \neq 0$ in \mathcal{U} , Let

$$\tau(z) = \frac{\dot{\omega}(\mu) - i\dot{\omega}(i\mu)}{\omega_z}.$$

Note that ω is real, thus

$$\bar{\tau}(z) = \frac{\dot{\omega}(\mu) + i\dot{\omega}(i\mu)}{\omega_{\bar{z}}}.$$

Since $\omega_{z\bar{z}} = 0$, we get

$$\tau_{\bar{z}} = \frac{\dot{\omega}_{\bar{z}}(\mu) - i\dot{\omega}_{\bar{z}}(i\mu)}{\omega_z}, \bar{\tau}_z = \frac{\dot{\omega}_{\bar{z}}(\mu) + i\dot{\omega}_{\bar{z}}(i\mu)}{\omega_{\bar{z}}},$$

The equality of (*) holds if and only if

$$\tau_{\bar{z}}\omega_z = f(z)\bar{\mu}\omega_z, \bar{\tau}_z\omega_{\bar{z}} = g(z)\bar{\mu}\omega_{\bar{z}},$$

where $f(z)$ and $g(z)$ are real valued functions defined on \mathcal{U} .

Since $\omega_z \neq 0$ and $\omega_{\bar{z}} \neq 0$, we have $\tau_{\bar{z}} = f(z)\bar{\mu}$, $\bar{\tau}_z = g(z)\bar{\mu}$.

We claim that $\tau_{\bar{z}} = 0$ in \mathcal{U} . In fact,

$$\tau_{\bar{z}} = f(z)\bar{\mu} = g(z)\mu.$$

By assumption, $\mu = \frac{\Phi}{g_0}$ where Φ is a holomorphic quadratic differential. If z is a point in \mathcal{U} such that $f(z) \neq 0$ (and then $g(z) \neq 0$), then we have

$$\frac{\Phi^2(z)}{|\Phi(z)|^2} = \frac{\Phi(z)}{\bar{\Phi}(z)} = \frac{g(z)}{f(z)}.$$

It turns out that z should be a zeros of Φ . Otherwise, the value of $\Phi^2(z)$ is real in a neighborhood of z , which is impossible. As a result, $\tau_{\bar{z}} = 0$ in \mathcal{U} .

It follows that

$$\dot{\omega}_{\bar{z}}(\mu) - i\dot{\omega}_{\bar{z}}(i\mu) = 0, \dot{\omega}_{\bar{z}}(\mu) + i\dot{\omega}_{\bar{z}}(i\mu) = 0.$$

In this case strict inequality

$$\int_S |\dot{\omega}_z(\mu)|^2 dz \bar{z} + \int_S |\dot{\omega}_z(i\mu)|^2 dz \bar{z} < 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z}$$

is trivial.

Putting all these together we obtain that

$$\int_S |\dot{\omega}_z(\mu)|^2 dz \bar{z} + \int_S |\dot{\omega}_z(i\mu)|^2 dz \bar{z} < 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z}.$$

The strictly pluri-subharmonicity holds. \square

5.3. An inequality. In the following, we apply (12) to present an inequality about the second variation of $E(\omega^t, g_t) = Ext_\gamma(g_t)$ along a Teichmüller geodesic.

Proposition 5.2. *Let (S, g_t) be a differential family of hyperbolic structures in $\mathcal{T}(S)$ with Beltrami differential $\mu(t) = t\mu + o(t^2)$ at $t = 0$ and $\|\mu\|_\infty = 1$. Then the following inequality holds:*

$$\begin{aligned} \frac{d^2}{dt^2}|_{t=0} E(\omega^t, g_t) &\geq -2 \int_S (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} \\ &= -4 \int_S |\omega_z|^2 dz d\bar{z} = -4E(\omega, g_0). \end{aligned}$$

Proof. As above, we assume that ω is real and then $\dot{\omega}_z = \bar{\omega}_z$. By (12), we have

$$\begin{aligned} \int |\dot{\omega}_z|^2 dz d\bar{z} &= \left| \int_S \bar{\dot{\omega}}_z \mu \omega_z + \int_S \bar{\dot{\omega}}_z \bar{\mu} \omega_{\bar{z}} dz d\bar{z} \right| \\ &\leq \int_S |\bar{\dot{\omega}}_z \mu \omega_z| dz d\bar{z} + \int_S |\bar{\dot{\omega}}_z \bar{\mu} \omega_{\bar{z}}| dz d\bar{z} \\ &= \int_S |\dot{\omega}_z| [\mu(|\omega_z| + |\omega_{\bar{z}}|)] dz d\bar{z} \\ &\leq \frac{1}{2} \int_S |\dot{\omega}_z|^2 dz d\bar{z} + \frac{1}{2} \int_S |\mu|^2 (|\omega_z| + |\omega_{\bar{z}}|)^2 dz d\bar{z}. \end{aligned}$$

Then

$$\frac{1}{2} \int |\dot{\omega}_z|^2 dz d\bar{z} \leq \frac{1}{2} \int |\mu|^2 (|\omega_z| + |\omega_{\bar{z}}|)^2 dz d\bar{z}.$$

Equivalently,

$$\int_S |\dot{\omega}_z|^2 dz d\bar{z} \leq \int_S |\mu|^2 (|\omega_z| + |\omega_{\bar{z}}|)^2 dz d\bar{z}.$$

As the image of ω is \mathbb{R} -tree, the Jacobian is zero. We have

$$0 = |\omega_z|^2 - |\omega_{\bar{z}}|^2.$$

Then

$$\begin{aligned} \int_S |\dot{\omega}_z|^2 dz d\bar{z} &\leq \int_S |\mu|^2 (2|\omega_z|)^2 dz d\bar{z} \\ &\leq 4 \int_S |\mu|^2 |\omega_z|^2 dz d\bar{z} \\ &= 2 \int_S |\mu|^2 (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z}. \end{aligned}$$

Combining the above inequality with Theorem 3.4, we have

$$\begin{aligned}
 \frac{d^2}{dt^2}|_{t=0}E(\omega^t, g_t) &= \frac{d^2}{dt^2}|_{t=0}E(\omega, g_t) - \frac{d^2}{dt^2}|_{t=0}E(\omega^t, g_0) \\
 &= 2 \int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} - 2 \int_S |\dot{\omega}_z|^2 dz d\bar{z} \\
 &\geq 2 \int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} - 4 \int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} \\
 &= -2 \int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z}.
 \end{aligned}$$

By assumption $\|\mu\|_\infty = 1$, thus

$$\frac{d^2}{dt^2}|_{t=0}E(\omega^t, g_t) \geq -2 \int_S (|\omega_z|^2 + |\omega_{\bar{z}}|^2) dz d\bar{z} = -4E(\omega, g_0).$$

□

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